STATS 369 Homework 4

October 22, 2021

Exercise 1.10

(a) See Figure 1 for $T_1 = 0.1$ and Figure 2 for $T_2 = 0.6$.

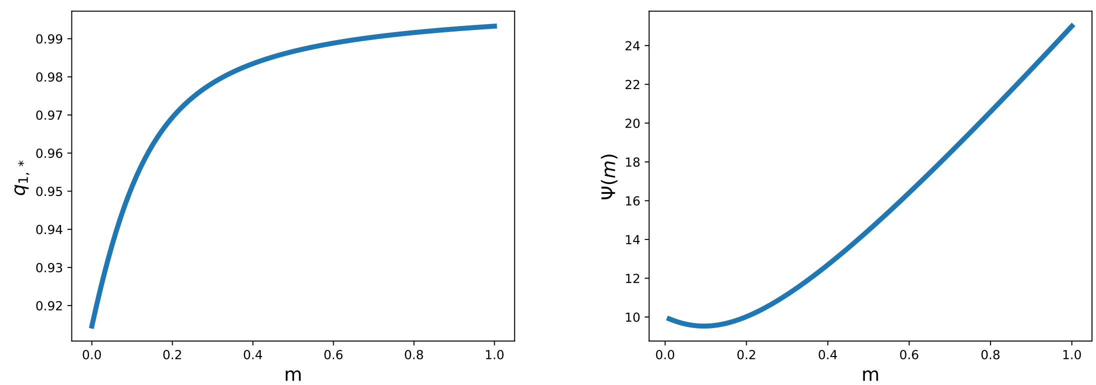


Figure 1: T_1

(b) We have

$$\begin{aligned}\frac{d}{dm}[m\Psi(q_{1,*}(m), m)] &= \Psi(q_{1,*}(m), m) + m\frac{d}{dm}\Psi(q_{1,*}(m), m) \\ &= \Psi(q_{1,*}(m), m) + m[\frac{d\Psi}{dq_1}\frac{dq_{1,*}(m)}{dm} + \frac{d\Psi}{dm}]. \end{aligned} \tag{1}$$

Since $q_{1,*}(m)$ is the solution of $f_1(q_1) = 2T^2/k$, by implicit function theorem, we have

$$\begin{aligned}\frac{dq_{1,*}(m)}{dm} &= -\frac{q_1^{k-1}(1-q_1)}{q_1^{k-2}(1-q_1)(1-(1-m)q_1)(\frac{k-2}{q_1} - \frac{1}{1-q_1} - \frac{1-m}{1-(1-m)q_1})} \\ &= \frac{q_1}{(1-(1-m)q_1)(\frac{1}{q_1} - \frac{1}{1-q_1} - \frac{1-m}{1-(1-m)q_1})}. \end{aligned} \tag{2}$$

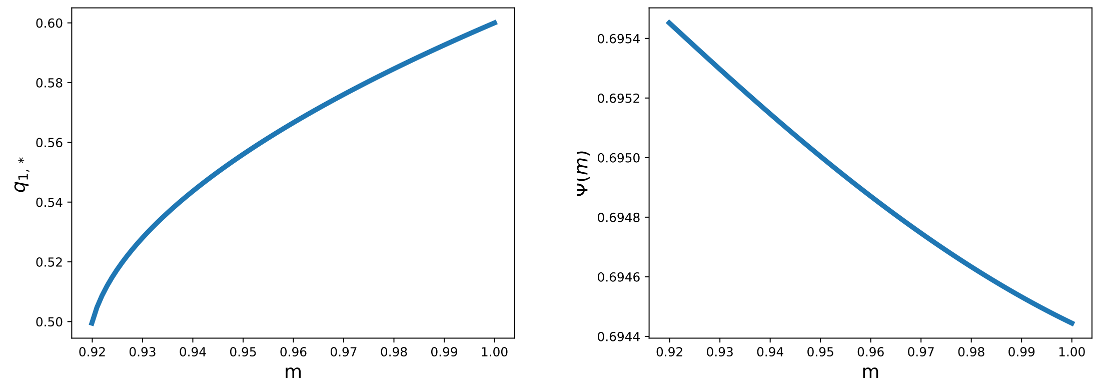


Figure 2: T_2

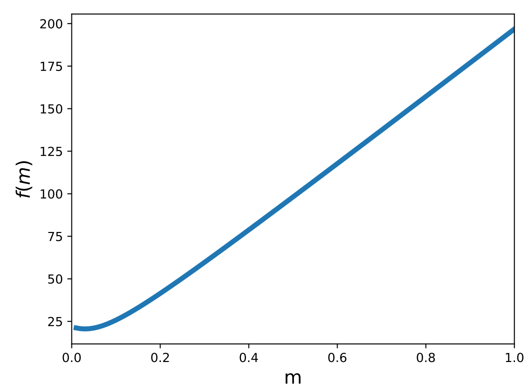
Moreover,

$$\frac{d\Psi}{dq_1} = -\frac{\beta^2}{4}(1-m)3q^2 + \frac{1-m}{2}\frac{q_1}{(1-(1-m)q_1)(1-q_1)}, \tag{3}$$

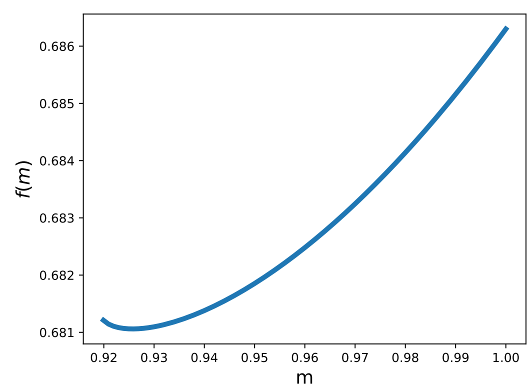
$$\frac{d\Psi}{dm} = \frac{\beta^2}{4}q_1^3 - \frac{1}{2m^2}\log\frac{1-(1-m)q_1}{1-q_1} + \frac{1}{2m}\frac{q_1}{1-(1-m)q_1}. \tag{4}$$

Substituting Equations (2), (3), and (4) into Equation (1) gives the answer.

The plot is in Figure 3.



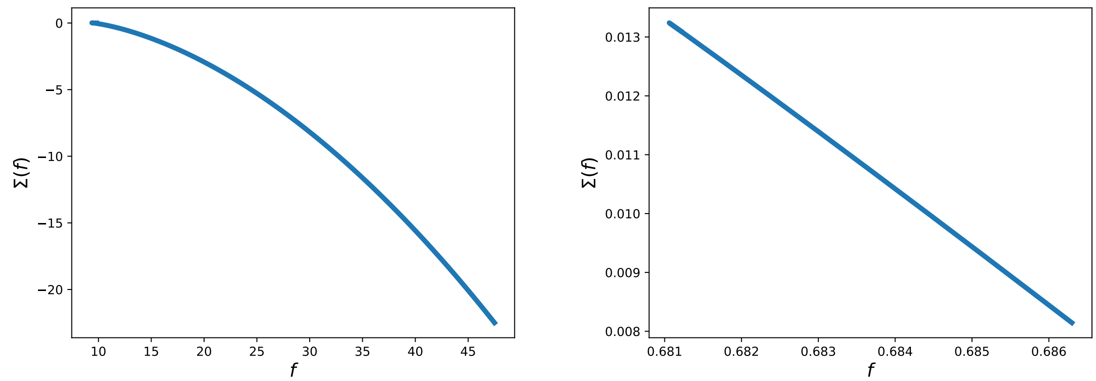
(a) T_1



(b) T_2

Figure 3: $f(m)$ versus m

(c) See Figure 4



(a) T_1 (b) T_2

Figure 4: $\Sigma(f)$ versus f

(d) For $T = T_1$, we have $f_d = 9.529110$, $f_s = 9.979164$, $\Sigma'(f_s) = -0.001002 > -1$, $f_* = 9.529110$.

For $T = T_2$, we have $f_d = 0.681211$, $f_s = 0.686299$, $\sigma'(f_s) = -1.0000004 < -1$, $f_* = 0.686299$.

For $T < T_s$, $\Sigma_\beta \geq 0$ if and only if $\Psi'(m) \leq 0$, and equivalently, $m \leq m_*$. So for $T < T_s$, $f_d = f(0)$, $f_s = f_* = f(m_*) = \Psi(m_*)$

For $T_s < T < T_d$, $f_* = f_s = f(1)$

(e) See Figure 5.

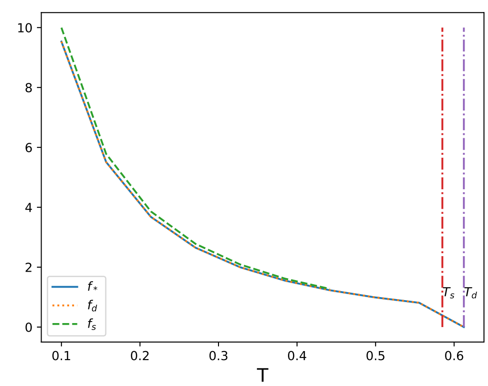


Figure 5: $f(T)$ versus T.